ME-221
SOLUTIONS FOR PROBLEM SET 1

Problem 1

a) A system is considered linear if the superposition principle holds.

For $u_1 \to y_1$ we obtain $\ddot{y}_1 + 3\dot{y}_1 + 2y_1 = u_1$, $y_1(0) = \dot{y}_1(0) = 0$
For $u_2 \to y_2$ we obtain $\ddot{y}_2 + 3\dot{y}_2 + 2y_2 = u_2$, $y_2(0) = \dot{y}_2(0) = 0$

The sum of these two equations is given by $(\ddot{y}_1 + \ddot{y}_2) + 3(\dot{y}_1 + \dot{y}_2) + 2(y_1 + y_2) = u_1 + u_2$, which is equal to $\ddot{y} + 3\dot{y} + 2y = u$ with $y = y_1 + y_2$ and $u = u_1 + u_2$. Thus, the additivity property is satisfied.

For input au, we obtain:

$$\ddot{y_a} + 3\dot{y_a} + 2y_a = au \tag{1}$$

For input u, we obtain:

$$\ddot{y} + 3\dot{y} + 2y = u \tag{2}$$

By multiplying both sides of the second equation by a, we obtain:

$$a\ddot{y} + 3a\dot{y} + 2ay = au \tag{3}$$

For $y_a = ay$, we have $\dot{y_a} = a\dot{y}$ and $\ddot{y_a} = a\ddot{y}$. By substituting in equation 1, we get equation 3. Therefore, for a scalar a, the output of the system to the input au is given by ay as they satisfy the differential equation together. Thus, the homogeneity property is satisfied. As a result (additivity + homogeneity), the system is linear.

b) The model of the system is considered to be non-linear when the variables and/or their derivatives have non-linear terms. For example, multiplying the input u with the output y ($\ddot{y} + 3\dot{y} + 2y = uy$) would make the mathematical model non-linear.

c) For non-zero initial conditions, the system is not 'initially at rest'. As a consequence, the response of the system at a given time depends on the input signal (forced response) and the initial conditions (natural response). If the model is linear, the principle of superposition applies to both homogenous and particular solutions. In other words, non-zero initial conditions do not affect the linearity of the system.

Problem 2

$$y(t) = F[x(t)] = \int_{-\infty}^{2t} u(\tau)d\tau$$

First, let's apply the delayed version of the input, $u(t - t_0)$ to the system.

$$F[x(t - t_0)] = \int_{-\infty}^{2t} u(\tau - t_0)d\tau$$

Denote $s = \tau - t_0$ for change of variables as follows

$$F[x(t-t_0)] = \int_{-\infty}^{2t-t_0} u(s)ds$$

Now, delaying the output by t_0 gives us

$$y(t-t_0) = \int_{-\infty}^{2(t-t_0)} u(\tau)d\tau = \int_{-\infty}^{2t-2t_0} u(\tau)d\tau$$

From these two equations we can see that $F[x(t-t_0)] \neq y(t-t_0)$. Thus, the given system is time-variant.

Problem 3

The system is linear because:

- Differentiation is a linear operator. Therefore, the derivative of y is not an issue (bonus: the presence of a derivative makes the system dynamic).
- It is independent of the coefficients. Therefore, the 2(t+1) term does not impact linearity.
- There are no other particularities to the system, so it is linear.

It is time-variant because:

- The 2(t+1) coefficient depends on time; therefore, it is time-variant (not time-invariant).

It is causal because:

- What happens at t does not depend on inputs from the future u(t+a), where a¿0.

b) A non-causal version of the system is $\dot{y}(t) = 2ty(t) + 3u(t+2)$. The implication of this modification is that a future entry would act on the present output of the system, which is not currently possible for physical systems. Advancements in quantum information processing systems may relax this fundamental limitation.

Problem 4

If you take a careful look at the signals, you will see that $u_2(t) = u_1(t) + u_1(t+1)$.

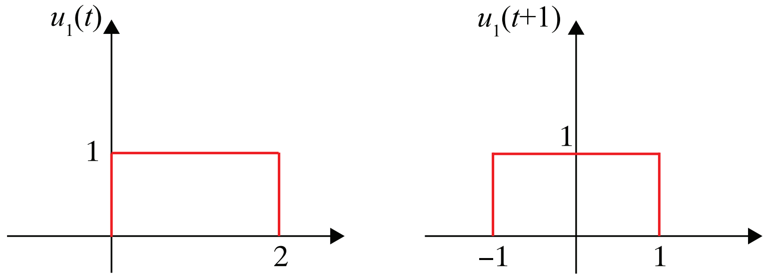

Figure 1: Time delayed version of input $u_1(t)$.

The system is time-invariant, thus the output of the system to the input $u_1(t+1)$ is $y_1(t+1)$. Furthermore, the output of the system to the input $u_2(t) = u_1(t) + u_1(t+1)$ can be found as $y_2(t) = y_1(t) + y_1(t+1)$ (due to linearity).